On the Consistent Effect Histories Approach to Quantum Mechanics

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ABSTRACT

A formulation of the consistent histories approach to quantum mechanics in terms of generalized observables (POV measures) and effect operators is provided. The usual notion of 'history' is generalized to the notion of 'effect history'. The space of effect histories carries the structure of a D-poset. Recent results of J.D. Maitland Wright imply that every decoherence functional defined for ordinary histories can be uniquely extended to a bi-additive decoherence functional on the space of effect histories. Omnès' logical interpretation is generalized to the present context. The result of this work considerably generalizes and simplifies the earlier formulation of the consistent effect histories approach to quantum mechanics communicated in a previous work of this author.

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I. Introduction

Nonrelativistic quantum mechanics in its standard formulation is not a theory which describes dynamical processes in time, but it is a theory which gives probabilities to various possible events and measurement outcomes at fixed instants of time. The dynamical law of quantum mechanics, the Schrödinger equation, describes the change of the probability amplitude with time. Quantum mechanics in its usual form does not provide us with a dynamical law which describes the time evolution of events. This can be succinctly summarized by saying that quantum mechanics in its usual form does not provide us with a (naive) model what is "actually" going on on a microscopic level in a quantum system. It is often felt that this is a serious drawback of quantum mechanics. Examples for attempts to modify quantum mechanics to a theory providing us with a model for what is "actually" happening are hidden variables theories (see, e.g., Refs. [1], [2], [3]), the dynamical state vector reduction models (see, e.g., Refs. [4, 5, 6] and [7]) or related models (see, e.g., Ref. [8]).

The consistent histories formulation of quantum mechanics is another attempt to remedy the situation and to incorporate time sequences of events and – as a special case – sequential measurements into quantum mechanics without providing a naive dynamical model for the microscopic world in the above sense and without altering the basic principles and the basic mathematical structure of Hilbert space quantum mechanics.

The consistent histories approach to nonrelativistic quantum mechanics has been inaugurated in a seminal paper by Griffiths [9] and further developed by Griffiths [10, 11, 12], by Omnès [13]-[20], by Isham [21], Isham and Linden [22, 23] and by Isham, Linden and Schreckenberg [24] and applied to quantum cosmology by Gell-Mann and Hartle [25]-[30] and Hartle [31, 32]. Dowker and Kent have carried out a critical reexamination of the consistent histories approach and particularly of Omnès' notion of truth and of the Gell-Mann–Hartle programme, see Refs. [33]-[35]. A critical discussion of the consistent histories approach can also be found in Ref. [36]. The consistent histories approach asserts that quantum mechanics provides a realistic description of individual quantum mechanical systems, regardless of whether they are open or closed. The possibility of a quantum mechanical description of single closed systems, which do neither interact with their environment nor are exposed to measurements, is denied by the conventional Copenhagen-type interpretations of quantum mechanics.

On the contrary, in the logical interpretation developed by Omnès the notion of measurement is not a key concept. Instead one takes the point of view that the aim of an interpretation is generally to provide us with a systematic and unambiguous language specifying the meaning of the objects in the formalism in terms of real physical objects and specifying what can meaningfully be said about the physical systems described by the theory. We will call this attitude the *semantic* approach to interpretation. Clearly the logical interpretation is a realistic interpretation in the sense that it is presupposed that physical systems really exist and have real properties regardless of whether they are measured or not.

A key notion in the formulation of quantum mechanics is the notion of observable. In the spirit of the logical interpretation the term *speakable* would be more appropriate, but we stick to the usual terminology. In usual Hilbert space quantum mechanics the observables are identified with self-adjoint operators on the Hilbert space and propositions about quantum mechanical

systems are identified with projection operators on Hilbert space. There is a one-one correspondence between self-adjoint operators on Hilbert space and projection valued (PV) measures on the real line $\mathbb{R}$. To every Borel subset $\mathcal{B}$ of $\mathbb{R}$ there corresponds one projection operator representing the proposition that the value of the considered observable is in the set $\mathcal{B}$. Some more remarks about observables and propositions in ordinary quantum mechanics can be found in Ref. [37].

The question which objects in the formalism have to be identified with observables (or speakables) is clearly a question belonging to the interpretation of quantum mechanics. Reasonableness and mathematical simplicity are the guiding principles to answer this question. The most general notion of observable compatible with the probabilistic structure of quantum mechanics is that of *positive operator valued (POV) measures* which contains the ordinary observables represented by PV measures on $\mathbb{R}$ as a subclass. Quantum mechanics is totally consistent without POV measures, but POV measures enrich the language of quantum mechanics and enlarge the measurement theoretical possibilities of quantum mechanics [38, 39]. On the other hand the claim that POV measures represent the observables in quantum mechanics is not only consistent with the mathematical structure of Hilbert space quantum mechanics but furthermore is also reasonable. Many examples can be found in the monograph by Busch et al. [40].

In this work we take on the view that POV measures are the observables in quantum mechanics and that all POV measures should be treated on the same footing and that all effects should be identified with the general properties (or speakables or beables) of quantum systems. We further consider *every* effect operator as representative of some sort of reality. Some arguments supporting this view can be found in Ref. [37] and references therein. It is perhaps worthwhile to mention a further simple argument which is essentially due to Ludwig [41]. To this end consider a measuring device $\mathcal{M}$ consisting of a detector $\mathcal{D}$ (designed to measure some property E associated with some projection operator) and some scatterer $\mathcal{S}$. An appropriately prepared incident physical system $\mathcal{I}$ (e.g., a particle) is first scattered by $\mathcal{S}$ and then detected by the detector $\mathcal{D}$. To obtain the property F measured by the device $\mathcal{M}$ one has to apply the unitary transformation given by the S-matrix S of $\mathcal{S}$ to the property measured by $\mathcal{D}$. Let $\varrho_{\mathcal{I}}$ denote the initial state of $\mathcal{I}$ and $\varrho_{\mathcal{S}}$ denote the initial state of $\mathcal{S}$. Then the relation between E and F is given by

$$\mathrm{tr}\left(S(\varrho_{\mathcal{I}} \otimes \varrho_{\mathcal{S}})S^{\dagger}(1 \otimes E)\right) = \mathrm{tr}\left(\varrho_{\mathcal{I}} F\right),$$

where the trace on the right hand side is in the Hilbert space $\mathbb{H}_{\mathcal{I}}$ of $\mathcal{I}$ and the trace on the left hand side is in the tensor product $\mathbb{H}_{\mathcal{I}} \otimes \mathbb{H}_{\mathcal{S}}$ of the Hilbert spaces $\mathbb{H}_{\mathcal{I}}$ of $\mathcal{I}$ and $\mathbb{H}_{\mathcal{S}}$ of $\mathcal{S}$. The operator F is uniquely determined by this equation. However, realistic physical S-matrices S transform projection operators (according to the above equation) in general to effect operators and only the set of effect operators is invariant under this transformations. Therefore whether a measuring device measures an effect or a property associated with some projection operator may depend on an arbitrary cut between the system and the apparatus. This argument can be formalized, see Ref. [42].

In the consistent histories approach it is claimed that all results of measurement theory also *follow* from the consistent histories approach. In the present work we take seriously this claim and continue our efforts to formulate the consistent histories formalism for general observables represented by POV measures. This programme has first been formulated and studied in

Ref. [37]. We will freely use the notation and terminology from Ref. [37] and review only the bare essentials.

This work is organized as follows: In Section II we summarize the consistent histories approach to nonrelativistic Hilbert space quantum mechanics and the logical interpretation of quantum mechanics. In Section III we recall basic definitions and results from Ref. [37] and we formulate our generalized (effect) history theory and a generalized logical rule of interpretation for effect histories. Our results are based on an important theorem by Wright [43]. This theorem relies heavily on the recent solution of the Mackey-Gleason problem, see Refs. [44, 45]. The results in Section III considerably simplify and generalize the results formulated in Ref. [37]. Section IV presents our summary.

As in Ref. [37] it must be emphasized that the representation and the interpretation of the consistent histories approach in this work might not be accepted by the authors cited. The present work solely reflects the inclination and the views of this author.

II.   Consistent Histories and the Logical Interpretation

We consider a quantum mechanical system $\mathcal{S}$ without superselection rules represented by a separable complex Hilbert space $\mathbb{H}$ and a Hamiltonian operator H. Every physical state of the considered system is mathematically represented by a density operator on $\mathbb{H}$, i.e., a linear, positive, trace-class operator on $\mathbb{H}$ with trace 1. The time evolution is governed by the unitary operator $U(t', t) = \exp(-i(t' - t)H/\hbar)$ which maps states at time t into states at time t' and satisfies $U(t'', t')U(t', t) = U(t'', t)$ and $U(t, t) = 1$.

In the familiar formulations of quantum mechanics the observables are identified with (and represented by) the self-adjoint operators on $\mathbb{H}$ and according to the spectral theorem observables can be identified with projection operator valued (PV) measures on the real line; that is, there is a one-to-one correspondence between self-adjoint operators on $\mathbb{H}$ and maps $\mathcal{O} : \mathcal{B}(\mathbb{R}) \to \mathcal{P}(\mathbb{H})$, such that $\mathcal{O}(\mathbb{R}) = 1$ and $\mathcal{O}(\cup_i K_i) = \sum_i \mathcal{O}(K_i)$ for every pairwise disjoint sequence $\{K_i\}_i$ in $\mathcal{B}(\mathbb{R})$ (the series converging in the ultraweak topology). Here $\mathcal{B}(\mathbb{R})$ denotes the Borel σ-algebra of $\mathbb{R}$ and $\mathcal{P}(\mathbb{H})$ denotes the set of projection operators on $\mathbb{H}$, i.e., self-adjoint operators P satisfying $P = PP$.

A meaningful proposition about the system (also called *physical quality*) is a proposition specifying that the value of some observable $\mathcal{O}$ lies in some set $B \in \mathcal{B}(\mathbb{R})$. This means that to every meaningful proposition about the system under consideration there corresponds one projection operator on $\mathbb{H}$.

In the state represented by the density operator ϱ the probability of a proposition represented by the projection operator P is given by $\text{tr}(\varrho P)$, where tr denotes the trace in $\mathbb{H}$.

Positive and bounded operators F on $\mathbb{H}$, satisfying $0 \leq F \leq 1$, are commonly called EFFECTS and the set of all effects on the Hilbert space $\mathbb{H}$ will be denoted by $\mathfrak{E}(\mathbb{H})$. We further denote the set of all bounded, linear operators on $\mathbb{H}$ by $\mathcal{B}(\mathbb{H})$.

If $\mathbb{H}$ is an infinite dimensional Hilbert space, then the set of all projection operators $\mathcal{P}(\mathbb{H})$ on $\mathbb{H}$ is weakly dense in $\mathfrak{E}(\mathbb{H})$ [46].

GENERALIZED OBSERVABLES are now identified with positive operator valued (POV) measures on some measurable space $(\Omega, \mathcal{F})$, i.e., maps $\mathsf{O} : \mathcal{F} \to \mathfrak{E}(\mathbb{H})$ with the properties:

- $\mathsf{O}(A) \geq \mathsf{O}(\emptyset)$, for all $A \in \mathcal{F}$;

- Let $\{A_i\}$ be a countable set of disjoint sets in $\mathcal{F}$, then $\mathsf{O}(\cup_i A_i) = \sum_i \mathsf{O}(A_i)$, the series converging ultraweakly;

- $\mathsf{O}(\Omega) = 1$.

Generalized observables are also called EFFECT VALUED MEASURES. Ordinary observables (associated with self-adjoint operators on $\mathbb{H}$) are then identified with the projection valued measures on the real line $\mathbb{R}$. Generalizing our above terminology, we regard all propositions specifying the value of some generalized observable as generalized physical qualities . In order to discriminate physical qualities corresponding to ordinary observables from physical qualities corresponding to generalized observables, we will call the former 'ordinary physical qualities' and the latter 'generalized physical qualities.' In the generalized approach to every physical quality there corresponds one effect operator.

A HOMOGENEOUS HISTORY is a map $h : \mathbb{R} \to \mathcal{P}(\mathbb{H}), t \mapsto h_t$. We call $t_i(h) := \min(t \in \mathbb{R} \mid h_t \neq 1)$ the INITIAL and $t_f(h) := \max(t \in \mathbb{R} \mid h_t \neq 1)$ the FINAL TIME of h respectively. Furthermore, the SUPPORT OF h is given by $\mathfrak{s}(h) := \{t \in \mathbb{R} \mid h_t \neq 1\}$. If $\mathfrak{s}(h)$ is finite, countable or uncountable, then we say that h is a FINITE, COUNTABLE or UNCOUNTABLE HISTORY respectively. The space of all homogeneous histories will be denoted by $\mathcal{H}(\mathbb{H})$, the space of all finite homogeneous histories by $\mathcal{H}_{fin}(\mathbb{H})$ and the space of all finite homogeneous histories with support S by $\mathcal{H}_S(\mathbb{H})$.

By a *history proposition* we mean a proposition about the system specifying which history will be realized. We use the terms history and history proposition synonymously in this work.

In this work we focus attention on finite histories. If a homogeneous history vanishes for some $t_0 \in \mathbb{R}$, i.e., $h_{t_0} = 0$, then we say that h is a ZERO HISTORY. All zero histories are collectively denoted by 0, slightly abusing the notation.

For every finite subset S of $\mathbb{R}$ we can consider the Hilbert tensor product $\otimes_{t \in S}\mathbb{H}$ and the algebra $\mathcal{B}_S^\otimes(\mathbb{H}) := \mathcal{B}(\otimes_{t \in S}\mathbb{H})$ of bounded linear operators on $\otimes_{t \in S}\mathbb{H}$. It has been pointed out by Isham [21] that for any fixed S there is an injective (but not surjective) correspondence σ_S between finite histories with support S and elements of $\mathcal{B}_S^\otimes(\mathbb{H})$ given by

$$\sigma_S : \mathcal{H}_S(\mathbb{H}) \to \mathcal{B}_S^\otimes(\mathbb{H}), h \simeq \{h_{t_k}\}_{t_k \in S} \mapsto \otimes_{t_k \in S} h_{t_k}. \tag{1}$$

The finite homogeneous histories with support S can therefore be identified with projection operators on $\otimes_{t \in S}\mathbb{H}$. The set of all projection operators on $\otimes_{t \in S}\mathbb{H}$ will in the sequel be denoted by $\mathcal{P}_S^\otimes(\mathbb{H})$. However, not all projection operators in $\mathcal{P}_S^\otimes(\mathbb{H})$ have the form $\sigma_S(h)$ with $h \in \mathcal{H}_S(\mathbb{H})$. The projection operators in $\mathcal{P}_S^\otimes(\mathbb{H})$ are called FINITE INHOMOGENEOUS HISTORIES WITH SUPPORT S and the space $\mathcal{K}_S(\mathbb{H}) := \mathcal{P}_S^\otimes(\mathbb{H})$ of projection operators on $\otimes_{t \in S}\mathbb{H}$ the SPACE OF FINITE INHOMOGENEOUS HISTORIES WITH SUPPORT S. The space of all finite inhomogeneous histories with arbitrary support will be denoted by $\mathcal{K}_{fin}(\mathbb{H})$ or by $\mathcal{P}_{fin}^\otimes(\mathbb{H})$. Furthermore, to every finite homogeneous history $h \in \mathcal{H}_{fin}(\mathbb{H})$ we associate its CLASS OPERATOR WITH RESPECT

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TO THE FIDUCIAL TIME t_0 by $C_{t_0}(h) := U(t_0, t_n) h_{t_n} U(t_n, t_{n-1}) h_{t_{n-1}} ... U(t_2, t_1) h_{t_1} U(t_1, t_0)$. The class operators can be unambiguously extended to finite inhomogeneous histories such that C_{t_0} is additive for orthogonal projectors, i.e., $C_{t_0}(h \vee k) := C_{t_0}(h) + C_{t_0}(k)$ for $h \perp k$. The functional $d_\varrho : \mathcal{K}_{fin}(\mathbb{H}) \times \mathcal{K}_{fin}(\mathbb{H}) \to \mathbb{C}, (h, k) \mapsto d_\varrho(h, k) := \text{tr}\left(C_{t_0}(h) \varrho(t_0) C_{t_0}(k)^\dagger \right)$ will be called the CONSISTENCY FUNCTIONAL ASSOCIATED WITH THE STATE ϱ. The consistency functional d_ϱ satisfies for all $h, h', k \in \mathcal{K}_{fin}(\mathbb{H})$

- $d_\varrho(h, h) \in \mathbb{R}$ and $d_\varrho(h, h) \geq 0$.

- $d_\varrho(h, k) = d_\varrho(k, h)^*$.

- $d_\varrho(1, 1) = 1$.

- $d_\varrho(h \vee h', k) = d_\varrho(h, k) + d_\varrho(h', k)$, whenever $h \perp h'$.

- $d_\varrho(0, h) = 0$, for all h.

In Ref. [37] we have used a slightly different terminology: the above consistency functional d_ϱ has been called there 'decoherence functional'. In this work we want to carefully distinguish bi-additive functionals on a Boolean lattice from bi-additive functionals defined on a D-poset. Thus, the former are called consistency functionals, whereas we reserve the term 'decoherence functional' for a bi-additive functional defined on a D-poset (see below).

Any collection $\mathcal{C}'$ of histories in $\mathcal{P}_{fin}^\otimes(\mathbb{H})$ is said to be CONSISTENT WITH RESPECT TO THE STATE ϱ if $\mathcal{C}'$ is a Boolean algebra (with respect to the meet, join and orthocomplementation in $\mathcal{P}_{fin}^\otimes(\mathbb{H})$ and with unit $1_{\mathcal{C}'}$) and if Re $d_\varrho(h, k) = 0$ for every two disjoint histories $h, k \in \mathcal{C}'$. Here two (possibly inhomogeneous) finite histories h and k are said to be DISJOINT if $h \leq \neg k$, where $\leq$ is the partial order on $\mathcal{K}_{\mathfrak{s}(h) \cup \mathfrak{s}(k)}(\mathbb{H})$.

It is now easy to see that the consistency functional d_ϱ induces an additive probability measure p_ϱ on every consistent Boolean sublattice $\mathcal{C} \subset \mathcal{P}_{fin}^\otimes(\mathbb{H})$. The probability measure p_ϱ is defined by

$$p_\varrho : \mathcal{C} \to \mathbb{R}^+, p_\varrho(h) := \frac{d_\varrho(h, h)}{d_\varrho(1_\mathcal{C}, 1_\mathcal{C})}. \tag{2}$$

The probability measure p_ϱ on a consistent Boolean algebra $\mathcal{C}$ of history propositions induced by the consistency functional d_ϱ according to Equation 2 defines two logical relations in $\mathcal{C}$, namely an implication and an equivalence relation between histories. A history proposition h is said to IMPLY a history proposition k if the conditional probability $p_\varrho(k|h) \equiv \frac{p_\varrho(h \wedge_\mathcal{C} k)}{p_\varrho(h)}$ is well-defined and equal to one. Two history propositions h and k are said to be EQUIVALENT if h implies k and vice versa.

The universal rule of interpretation of quantum mechanics can now be formulated as

Rule 1 (Omnès) *Propositions about quantum mechanical systems should solely be expressed in terms of history propositions. Every description of an isolated quantum mechanical system should be expressed in terms of finite history propositions belonging to a common consistent Boolean algebra of histories. Every reasoning relating several propositions should be expressed in terms of the logical relations induced by the probability measure from Equation 2 in that Boolean algebra.*

III. Consistent Effect Histories

In Ref. [37] we have motivated and introduced the following notion of homogeneous effect history

Definition 1 *A* HOMOGENEOUS EFFECT HISTORY (OF THE FIRST KIND) *is a map* $u : \mathbb{R} \to \mathfrak{E}(\mathbb{H}), t \mapsto u_t$. *The* SUPPORT OF u *is given by* $\mathfrak{s}(u) := \{t \in \mathbb{R} \mid u_t \neq 1\}$. *If* $\mathfrak{s}(u)$ *is finite, countable or uncountable, then we say that* u *is a* FINITE, COUNTABLE *or* UNCOUNTABLE EFFECT HISTORY *respectively. The space of all homogeneous effect histories (of the first kind) will be denoted by* $\mathbb{E}(\mathbb{H})$, *the space of all finite homogeneous effect histories (of the first kind) by* $\mathbb{E}_{fin}(\mathbb{H})$ *and the space of all finite homogeneous effect histories (of the first kind) with support* S *by* $\mathbb{E}_S(\mathbb{H})$. *All homogeneous effect histories for which there exists at least one* $t \in \mathbb{R}$ *such that* $u_t = 0$ *are collectively denoted by* 0, *slightly abusing the notation.*

The class operator C_{t_0} defined above for finite ordinary homogeneous histories can be defined for homogeneous finite effect histories $u \in \mathbb{E}_{fin}(\mathbb{H})$

$$C_{t_0}(u) \quad := \quad U(t_0, t_n)\sqrt{u_{t_n}}U(t_n, t_{n-1})\sqrt{u_{t_{n-1}}}...U(t_2, t_1)\sqrt{u_{t_1}}U(t_1, t_0).$$

For every pair u and v of finite homogeneous effect histories (of the first kind) we define the DECOHERENCE WEIGHT of u and v by

$$d_\varrho(u, v) := \operatorname{tr}\left(C_{t_0}(u)\varrho(t_0)C_{t_0}(v)^\dagger\right).$$

The functional $d_\varrho : \mathbb{E}_{fin}(\mathbb{H}) \times \mathbb{E}_{fin}(\mathbb{H}) \to \mathbb{C}, (u, v) \mapsto d_\varrho(u, v)$ will be called the DECOHERENCE FUNCTIONAL ASSOCIATED WITH THE STATE ϱ.

The map σ_S given by Equation 1 can be extended to a map

$$\sigma_{fin} : \mathbb{E}_{fin}(\mathbb{H}) \to \mathcal{B}^{\otimes}_{fin}(\mathbb{H}), u \simeq \{u_{t_k}\}_{t_k \in \mathfrak{s}(u)} \mapsto \otimes_{t_k \in \mathfrak{s}(u)} u_{t_k}, \tag{3}$$

where $\mathcal{B}^{\otimes}_{fin}(\mathbb{H})$ denotes the disjoint union of all $\mathcal{B}^{\otimes}_S(\mathbb{H}), S \subset \mathbb{R}$ finite. The map σ_{fin} is neither injective nor surjective. However, $d_\varrho(u, v)$ depends on u and v only through $\sigma_{fin}(u)$ and $\sigma_{fin}(v)$. From a mathematical point of view it thus seems to be natural to define the notion of *inhomogeneous* effect history as follows:

Definition 2 *Let* S *be a finite subset of* $\mathbb{R}$, *then we call the space* $\mathfrak{E}^{\otimes}_S(\mathbb{H}) := \mathfrak{E}(\otimes_{t \in S}\mathbb{H})$ *of effect operators on* $\otimes_{t \in S}\mathbb{H}$ *the* SPACE OF FINITE INHOMOGENEOUS EFFECT HISTORIES WITH SUPPORT S. *The space of all finite inhomogeneous effect histories with arbitrary support will be denoted by* $\mathfrak{E}^{\otimes}_{fin}(\mathbb{H})$. *The elements in* $\mathfrak{E}^{\otimes}_{fin}(\mathbb{H})$ *will also be called* EFFECT HISTORY PROPOSITIONS.

The homogeneous elements in $\mathfrak{E}^{\otimes}_{fin}(\mathbb{H})$ represent equivalence classes of homogeneous effect histories. In this work we will carefully distinguish between homogeneous effect histories as defined in Definition 1 and homogeneous elements in $\mathfrak{E}^{\otimes}_{fin}(\mathbb{H})$. For clarity of exposition we will call the former *homogeneous effect history of the first kind* or (where no confusion can arise) simply homogeneous effect histories, whereas the latter will be called *homogeneous effect histories of*

the second kind.

In technical terms $\mathfrak{E}_{fin}^{\otimes}(\mathbb{H})$ is the direct limit of the directed system $\{\mathfrak{E}_S^{\otimes}(\mathbb{H}) \mid S \subset \mathbb{R} \text{ finite}\}$. All the $\mathfrak{E}_S^{\otimes}(\mathbb{H})$, $S \subset \mathbb{R}$ and $\mathfrak{E}_{fin}^{\otimes}(\mathbb{H})$ carry several distinct D-poset structures, as discussed in Ref. [37]. We will denote the partial addition on $\mathfrak{E}_{fin}^{\otimes}(\mathbb{H})$ by $\oplus$ and the partial substraction on $\mathfrak{E}_{fin}^{\otimes}(\mathbb{H})$ by $\ominus$. For further literature on D-posets and effect algebras see Refs. [47]-[54]. As in Ref. [37] we use the terms *D-poset* and *effect algebra* synonymously. We refer to the D-poset structure on $\mathfrak{E}_{fin}^{\otimes}(\mathbb{H})$ given by the partial addition $\oplus$, where $E_1 \oplus E_2$ is defined if $E_1 + E_2 < 1$ by $E_1 \oplus E_2 := E_1 + E_2$, as the CANONICAL D-POSET STRUCTURE.

In Ref. [37] we have used a different notion of inhomogeneous effect history, because it was not clear whether the decoherence functional d_ϱ defined above on the space of homogeneous effect histories (of the first kind) can be (uniquely) extended to an in some appropriate sense bi-additive functional on the space of inhomogeneous effect histories as defined in Definition 2. In Ref. [37] we gave a rather technical definition the notion of inhomogeneous effect history. Essentially we defined an inhomogeneous effect history to be a member of the free lattice generated by the homogeneous effect histories (of the first kind) by at most finitely many applications of the grammatical connectives 'and' and 'or', to wit, we have viewed inhomogeneous effect histories to be – in essence – propositions in the language of quantum mechanics involving several (but at most finitely many) homogeneous effect histories. In turn only the latter were viewed as the basic *physical* entities in the formalism. We have shown in Ref. [37] that with this definitions it is possible to consistently extend the consistent histories formulation of quantum mechanics and to incorporate effect histories. However, this approach involves a rather technical and mathematically by no means canonical definition of the notion of inhomogeneous history. Inhomogeneous effect histories in the sense of Ref. [37] represent only semantical entities without an obvious physical interpretation. In this work the term inhomogeneous effect history is always meant in the sense of Definition 2 unless explicitly otherwise stated.

In this work we use a recent result of J.D. Maitland Wright [43] which implies that the decoherence functional d_ϱ as defined above on the space of homogeneous effect histories (of the second kind) can indeed be extended to a functional on the space of inhomogeneous effect histories with the desired properties. We first recall the central result from Ref. [43]

Theorem 1 *Let A be a von Neumann algebra with no type I_2 direct summand. Let $d : \mathcal{P}(A) \times \mathcal{P}(A) \to \mathbb{C}$ be a decoherence functional . If d is bounded, then d extends to a unique bounded bilinear functional $\widetilde{d}$ on $A \times A$. Furthermore, d is continuous when $\mathcal{P}(A)$ is equipped with the topology induced by the norm of A. Also, $d(u, v)^* = d(v^*, u^*)$.*

If A is a von Neumann algebra, let $\mathcal{P}(A)$ denote the set of projectors in A. A function $d : \mathcal{P}(A) \times \mathcal{P}(A) \to \mathbb{C}$ is called a DECOHERENCE FUNCTIONAL , if (i) $d(p_1 \oplus p_2, q) = d(p_1, q) + d(p_2, q)$, whenever p_1 and p_2 are mutually orthogonal; (ii) $d(p, q)^* = d(q, p)$; (iii) $d(p, p) \geq 0$; (iv) $d(1, 1) = 1$.

Since the set $\mathcal{B}(\mathbb{H})$ of bounded operators on a Hilbert space $\mathbb{H}$ with dimension greater than 2 is a von Neumann algebra (of type I), Theorem 1 can be applied to the decoherence functional $d_\varrho : \mathcal{K}_{fin}(\mathbb{H}) \times \mathcal{K}_{fin}(\mathbb{H}) \to \mathbb{C}, (h, k) \mapsto d_\varrho(h, k) := \text{tr}\left(C_{t_0}(h)\varrho(t_0)C_{t_0}(k)^\dagger\right)$ defined in Section II above. Thus for every finite subset $S \subset \mathbb{R}$ there is a unique bounded bilinear functional $\widetilde{d}_{\varrho,S}$

on $\mathcal{B}_S^{\otimes}(\mathbb{H}) \times \mathcal{B}_S^{\otimes}(\mathbb{H})$ extending the decoherence functional d_ϱ restricted to $\mathcal{P}_S^{\otimes}(\mathbb{H}) \times \mathcal{P}_S^{\otimes}(\mathbb{H})$.

The restriction $\widehat{d}_{\varrho,S}$ of $\widetilde{d}_{\varrho,S}$ to $\mathfrak{E}_S^{\otimes}(\mathbb{H}) \times \mathfrak{E}_S^{\otimes}(\mathbb{H})$ is a bounded functional which is additive in both arguments with respect to the canonical D-poset structure on $\mathfrak{E}_S^{\otimes}(\mathbb{H})$. The collection of all such functionals $\widehat{d}_{\varrho,S}$ for any finite $S \subset \mathbb{R}$ induces a bounded functional $\widehat{d}_\varrho$ on $\mathfrak{E}_{fin}^{\otimes}(\mathbb{H}) \times \mathfrak{E}_{fin}^{\otimes}(\mathbb{H})$ which is additive in both arguments with respect to the canonical D-poset structure on $\mathfrak{E}_{fin}^{\otimes}(\mathbb{H})$. The functional $\widehat{d}_\varrho$ will be called the DECOHERENCE FUNCTIONAL WITH RESPECT TO THE STATE ϱ ON $\mathfrak{E}_{fin}^{\otimes}(\mathbb{H})$.

Since $\mathfrak{E}_{fin}^{\otimes}(\mathbb{H})$ is a D-poset, $\mathfrak{E}_{fin}^{\otimes}(\mathbb{H})$ is in particular a partially ordered set. However, for two elements $e_1, e_2 \in \mathfrak{E}_{fin}^{\otimes}(\mathbb{H})$ the supremum $e_1 \vee e_2$ and the infimum $e_1 \wedge e_2$ not necessarily exist, that is, $\mathfrak{E}_{fin}^{\otimes}(\mathbb{H})$ is not a lattice. But there exists a partially defined join operation denoted by $\vee$ and a partially defined meet operation denoted by $\wedge$. To every element $e \in \mathfrak{E}_{fin}^{\otimes}(\mathbb{H})$ there exists one unique element $e' \in \mathfrak{E}_{fin}^{\otimes}(\mathbb{H})$ such that $e \oplus e'$ is well-defined and $e \oplus e' = 1$. We refer to $e' = 1 \ominus e$ as to the COMPLEMENT OF e.

Definition 3 *A subset $\mathcal{B} \subset \mathfrak{E}_{fin}^{\otimes}(\mathbb{H})$ is said to be an* ADMISSIBLE BOOLEAN LATTICE OF (INHOMOGENEOUS) EFFECT HISTORIES *if the following conditions are satisfied*

- *There exist two binary operations on $\mathcal{B}$, denoted by $\vee_\mathcal{B}$ and $\wedge_\mathcal{B}$ respectively, and one unary operation on $\mathcal{B}$, denoted by $\neg_\mathcal{B}$, such that the operations $\vee_\mathcal{B}, \wedge_\mathcal{B}$ and $\neg_\mathcal{B}$ are compatible with the partial order on $\mathcal{B}$ induced by the partial order on $\mathfrak{E}_{fin}^{\otimes}(\mathbb{H})$ and such that $(\mathcal{B}, \vee_\mathcal{B}, \wedge_\mathcal{B}, \neg_\mathcal{B})$ is a Boolean lattice. I.e., $\vee_\mathcal{B}$ is the join operation, $\wedge_\mathcal{B}$ is the meet operation and $\neg_\mathcal{B}$ is the complementation operation on $\mathcal{B}$; the lattice-operations $\vee_\mathcal{B}$ and $\wedge_\mathcal{B}$ coincide with the partially defined meet operation $\vee$ and join operation $\wedge$ on $\mathfrak{E}_{fin}^{\otimes}(\mathbb{H})$ whenever the latter are well-defined, to wit, $e_1 \wedge_\mathcal{B} e_2 = e_1 \wedge e_2$ and $e_3 \vee_\mathcal{B} e_4 = e_3 \vee e_4$ for all $e_1, e_2, e_3, e_4 \in \mathcal{B}$, whenever the right hand sides are well-defined in $\mathfrak{E}_{fin}^{\otimes}(\mathbb{H})$. The lattice-operations $\vee_\mathcal{B}$ and $\wedge_\mathcal{B}$ are such that a complementation $\neg_\mathcal{B}$ can be unambiguously defined on $\mathcal{B}$;*

- *There exists an injective map $\mathfrak{M} : \mathcal{B} \to \mathfrak{E}_{fin}^{\otimes}(\mathbb{H})$ which satisfies the following conditions*

    - *$\mathfrak{M}$ is a* positive valuation *on $\mathcal{B}$ with values in $\mathfrak{E}_{fin}^{\otimes}(\mathbb{H})$, to wit, a map satisfying the* valuation condition *$\mathfrak{M}(b_1 \vee_\mathcal{B} b_2) \ominus \mathfrak{M}(b_1) = \mathfrak{M}(b_2) \ominus \mathfrak{M}(b_1 \wedge_\mathcal{B} b_2)$, for all $b_1, b_2 \in \mathcal{B}$. This condition means in particular that the left hand side and the right hand side are well-defined for all $b_1, b_2 \in \mathcal{B}$;*

    - *$\mathfrak{M}$ preserves decoherence weights, i.e., $\widehat{d}_\varrho(e_1, e_2) = \widehat{d}_\varrho(\mathfrak{M}(e_1), \mathfrak{M}(e_2))$, for all $e_1, e_2 \in \mathcal{B}$.*

An admissible Boolean sublattice of $\mathfrak{E}_{fin}^{\otimes}(\mathbb{H})$ will be briefly denoted by $(\mathcal{B}, \mathfrak{M})$.

Remark 1 *Strictly speaking a* SUBLATTICE $\mathcal{L}$ OF $\mathfrak{E}_{fin}^{\otimes}(\mathbb{H})$ *is a subset $\mathcal{L} \subset \mathfrak{E}_{fin}^{\otimes}(\mathbb{H})$ such that $\mathcal{L}$ endowed with the restrictions of $\vee$ and $\wedge$ to $\mathcal{L}$ is a lattice. It makes thus sense to speak of sublattices of $\mathfrak{E}_{fin}^{\otimes}(\mathbb{H})$. However, it is important to notice that an admissible Boolean sublattice of $\mathfrak{E}_{fin}^{\otimes}(\mathbb{H})$ is not necessarily a sublattice of $\mathfrak{E}_{fin}^{\otimes}(\mathbb{H})$ in this sense.*

Remark 2 *Let $\mathcal{L}_1, \mathcal{L}_2$ be lattices. A map $\nu : \mathcal{L}_1 \to \mathcal{L}_2$ is called* POSITIVE *if $\nu(p) < \nu(q)$, whenever $p < q$. Since any Boolean lattice is relatively complemented, the condition that the map $\mathfrak{M}$ in Definition 3 is positive is actually redundant and follows already from the Definition of D-posets and from the valuation condition. In particular $\mathfrak{M}$ is order preserving.*

Remark 3 *The complement $e' = 1 - e$ in $\mathfrak{E}^{\otimes}_{fin}(\mathbb{H})$ of some element $e \in \mathcal{B}$ does in general not coincide with the complement $\neg_{\mathcal{B}} e$ in $\mathcal{B}$. The greatest element $1_{\mathcal{B}}$ and the least element $0_{\mathcal{B}}$ in $\mathcal{B}$ do not necessarily coincide with the greatest element 1 and the least element 0 in $\mathfrak{E}^{\otimes}_{fin}(\mathbb{H})$ respectively.*

Our target is to generalize Omnès' logical rule and thus to single out the appropriate subsets of $\mathfrak{E}^{\otimes}_{fin}(\mathbb{H})$ on which a reasoning involving (inhomogeneous) effect histories compatible with 'common sense' can be defined. The conditions in Definition 3 are clearly the minimal structure required. Usually 'common sense' (compare Ref. [19]) is tacitly associated with Boolean lattices. Thus the first condition in Definition 3 that $\mathcal{B}$ is a Boolean lattice is indispensable. We have already mentioned above that the set $\mathfrak{E}^{\otimes}_{fin}(\mathbb{H})$ carries (amongst others) a canonical D-poset structure, but no lattice structure and that the decoherence functional $\widehat{d}_{\varrho}$ is additive with respect to the D-poset structure on $\mathfrak{E}^{\otimes}_{fin}(\mathbb{H})$. In the consistent histories approach, however, reasoning is defined on Boolean lattices $\mathcal{B}$ with the help of consistency functionals which are additive with respect to the lattice structure of $\mathcal{B}$. Thus one has to restrict oneself to Boolean lattices $\mathcal{B} \subset \mathfrak{E}^{\otimes}_{fin}(\mathbb{H})$ such that the lattice structure of $\mathcal{B}$ is exactly mirrored in the D-poset structure of $\mathfrak{E}^{\otimes}_{fin}(\mathbb{H})$ (by the map $\mathfrak{M}$). This leads to the condition that there exists a positive valuation $\mathfrak{M}$ as required in the second condition of Definition 3. The reasoning to be defined should be independent of the map $\mathfrak{M}$ chosen. Thus it is necessary to require that $\mathfrak{M}$ preserves decoherence weights.

Remark 4 *The decoherence functional $\widehat{d}_{\varrho}$ induces a consistency functional $d_{\varrho,\mathcal{B}}$ on $\mathcal{B} \times \mathcal{B}$ by $d_{\varrho,\mathcal{B}} : \mathcal{B} \times \mathcal{B} \to \mathbb{C}, d_{\varrho,\mathcal{B}}(p_1, p_2) := \widehat{d}_{\varrho}(\mathfrak{M}(p_1), \mathfrak{M}(p_2))$, which is additive in both arguments with respect to the Boolean lattice structure on $\mathcal{B}$.*

Definition 4 *An admissible Boolean lattice $(\mathcal{B}, \mathfrak{M})$ is called* CONSISTENT W.R.T. *ϱ if for every pair of disjoint elements $b_1, b_2 \in \mathcal{B}$ (i.e., elements satisfying $b_1 \wedge_{\mathcal{B}} b_2 = 0$) the* `consistency condition` *$\operatorname{Re} d_{\varrho,\mathcal{B}}(b_1, b_2) = 0$ is satisfied.*

Theorem 2 *Let $(\mathcal{B}, \mathfrak{M})$ be a consistent admissible Boolean lattice of effect histories. Then the consistency functional $d_{\varrho,\mathcal{B}}$ induces a probability functional $p_{\varrho,\mathcal{B}}$ on $\mathcal{B}$ by $b \mapsto p_{\varrho,\mathcal{B}}(b) \equiv \frac{d_{\varrho,\mathcal{B}}(\mathfrak{M}(b), \mathfrak{M}(b))}{d_{\varrho,\mathcal{B}}(\mathfrak{M}(1_{\mathcal{B}}), \mathfrak{M}(1_{\mathcal{B}}))}$.*

Definition 5 *An effect history proposition $e_1 \in \mathfrak{E}^{\otimes}_{fin}(\mathbb{H})$ is said to* IMPLY *an effect history proposition $e_2 \in \mathfrak{E}^{\otimes}_{fin}(\mathbb{H})$ in the state ϱ if there exists a consistent admissible Boolean sublattice $\mathcal{B}$ of $\mathfrak{E}^{\otimes}_{fin}(\mathbb{H})$ containing e_1 and e_2 and if the conditional probability $p_{\varrho,\mathcal{B}}(e_2|e_1) \equiv \frac{p_{\varrho,\mathcal{B}}(e_1 \wedge_{\mathcal{B}} e_2)}{p_{\varrho,\mathcal{B}}(e_1)}$ is well-defined and equal to one. We write $e_1 \Longrightarrow_{\varrho} e_2$. Two history propositions e_1 and e_2 are said to be* EQUIVALENT *if e_1 implies e_2 and vice versa. We write $e_1 \Longleftrightarrow_{\varrho} e_2$.*

Remark 5 *If $e_1 \wedge e_2$ exists in $\mathfrak{E}^{\otimes}_{fin}(\mathbb{H})$, then it is easy to verify that if $p_{\varrho,\mathcal{B}_0}(e_2 \mid e_1)$ is well-defined and equal to one in some consistent admissible Boolean lattice $\mathcal{B}_0$ containing e_1 and e_2, then $p_{\varrho,\mathcal{B}}(e_2 \mid e_1)$ is well-defined and equal to one in every consistent admissible Boolean lattice $\mathcal{B}$ containing e_1 and e_2. If $e_1 \wedge e_2$ does not exists in $\mathfrak{E}^{\otimes}_{fin}(\mathbb{H})$, then there may be consistent Boolean lattices $\mathcal{B}_1$ containing e_1 and e_2 such that $p_{\varrho,\mathcal{B}_1}(e_2 \mid e_1)$ is not one or is not well-defined. If $e_1 \wedge e_2$ does not exists in $\mathfrak{E}^{\otimes}_{fin}(\mathbb{H})$, then it seems reasonable to define $e_1 \Longrightarrow_{\varrho} e_2$ if there exists an admissible Boolean lattice $\mathcal{B}$ containing e_1, e_2 and some further element $e_3 \in \mathfrak{E}^{\otimes}_{fin}(\mathbb{H})$ satisfying $e_1 \geq e_3$ and $e_2 \geq e_3$ such that $\frac{p_{\varrho,\mathcal{B}}(e_3, e_3)}{p_{\varrho,\mathcal{B}}(e_1, e_1)}$ is well-defined in $\mathcal{B}$ and equal to one.*

The generalized universal rule of interpretation of quantum mechanics can now simply be formulated as

Rule 3 *Propositions about quantum mechanical systems should solely be expressed in terms of effect history propositions. Every description of an isolated quantum mechanical system should be expressed in terms of finite effect history propositions belonging to a common consistent admissible Boolean algebra of effect histories. Every reasoning relating several propositions should be expressed in terms of the logical relations induced by the probability measure from Theorem 2 in that Boolean algebra.*

(This rule is numbered 'Rule 3' in order to distinguish it from Rule 2 stated in Ref. [37].) It is instructive to compare Rule 3 with Rule 2 stated in Ref. [37]. It is obvious that Rule 1 is contained in Rule 3 as a special case. A more extensive discussion of the motivation and the philosophy underlying the logical interpretation of quantum mechanics can be found in Refs. [17]-[19] and in Ref. [37] and will not be repeated here.
Compared with the treatment in Ref. [37] we have achieved a considerable simplification of the logical interpretation in terms of generalized observables and of the formalism of the consistent effect histories approach to generalized quantum mechanics. From a mathematical point of view, the extension of the ordinary consistent histories approach given in this article is a natural one. Rule 3 asserts that to every meaningful proposition about a quantum mechanical system there is an inhomogeneous effect history $e \in \mathfrak{E}^{\otimes}_{fin}(\mathbb{H})$. However, homogeneous effect histories of the first kind, which have a direct physical interpretation, are not contained in $\mathfrak{E}^{\otimes}_{fin}(\mathbb{H})$. According to Rule 3 homogeneous effect histories of the first kind can only indirectly be included into a description of a quantum mechanical system by representing every homogeneous effect history of the first kind e by its corresponding homogeneous effect history of the second kind $\sigma_{fin}(e)$.
It remains to determine the connection of Rule 3 stated above and the generalized logical rule (Rule 2) formulated in Ref. [37]. In contrast to Rule 3 above, the propositions about a quantum mechanical system permitted by Rule 2 stated in Ref. [37] contain the homogeneous effect histories of the first kind as a subclass and accordingly a description of a quantum mechanical system and reasoning can be done directly in terms of homogeneous effect histories of the first kind. In the next subsection we will see, however, that in an appropriate sense Rule 3 is a generalization of Rule 2 stated in Ref. [37] and that a description and reasoning (permitted by Rule 2) directly in terms of homogeneous effect histories of the first kind can always be lifted to a description and reasoning (permitted by Rule 3) in terms of the corresponding homogeneous effect histories of the second kind.

The connection between admissible and allowed Boolean lattices

In this subsection we will show that Rule 3 formulated above is indeed a generalization of the generalized logical rule as formulated in Ref. [37]. In this subsection we will use the notation and terminology introduced in Ref. [37] without further notice. In this subsection the term *homogeneous effect history* is always meant to denote homogeneous effect histories of the first kind.

Consider some homogeneous effect history of order $k > 0$ denoted by $w^k_{E_1,...,E_m}$, where $E_1, ..., E_m \in \mathfrak{E}(\mathbb{H})$. The corresponding history proposition states that first at k successive times $t_{1,1}, ..., t_{1,k}$ the appropriately time translated effect $E_1(t_{1,j}) = U(t_{1,j}, t_{1,1}) E_1 U(t_{1,j}, t_{1,1})^{\dagger}$ ($1 \leq j \leq k$) is realized and then at k successive times $t_{2,1}, ..., t_{2,k}$ the effect $E_2(t_{2,j}) = U(t_{2,j}, t_{1,1}) E_2 U(t_{2,j}, t_{1,1})^{\dagger}$ ($1 \leq j \leq k$) and so on (we refer the reader to the discussion following Theorem 4 in Ref. [37]; for simplicity we assume that the history w_0 appearing there is the unit history, i.e., $(w_0)_t = 1$ for all t).

Now we first observe that the exact times associated with the effects in some homogeneous effect history are inessential. The only thing that physically matters is the order and sequence of the effects in the homogeneous history. The time points associated with the effect operators in some homogeneous effect history can be changed provided the order remains fixed and provided the effect operators associated with the shifted times are appropriately time translated with the unitary evolution operator U. We say that two homogeneous effect histories related in this way to each other are SHIFT-EQUIVALENT.

If we define $F_j := E_j^{k/2}$, for all $1 \leq j \leq k$, then we see that every homogeneous effect history $w^k_{E_1,...,E_m}$ of order k can be mapped to a homogeneous effect history $w^2_{F_1,...,F_m}$ of order 2. This map preserves decoherence weights. The history $w^2_{F_1,...,F_m}$ is unique up to shift-equivalence. That the F_j are effect operators follows from Proposition 2 in Ref. [55]. We further recall that $F \oplus_1 F' = F \oplus F' = \left(E \oplus_{2/k} E' \right)^{k/2}$ where $F = E^{k/2}$ and $F' = (E')^{k/2}$ whenever the expressions are well-defined. Now it is easy to see that for every allowed Boolean algebra $(\mathcal{B}, \text{B})$ of order k (as defined in Ref. [37]) there exists an allowed Boolean algebra $(\mathcal{B}', \text{B}')$ of order 2 and a lattice isomorphism $\varphi : \mathcal{B} \to \mathcal{B}'$ such that $\text{B} = \text{B}' \circ \varphi$. Thus, it suffices to consider allowed Boolean algebras of order 2 in the sequel. In Theorem 3 below $\mathfrak{N}$ denotes the canonical map defined in Remark 15 in Ref. [37].

Theorem 3 *Let $(\mathcal{B}, \mathfrak{M})$ be an admissible Boolean lattice in the sense of Definition 3 above and let $(\mathcal{A}, \mathfrak{J})$ be an allowed Boolean lattice of effect histories of order k as defined in Ref. [37]. Let $\mathcal{A}_0$ denote the set of atoms of $\mathcal{A}$. Then there exists a lattice isomorphism $\psi : \mathcal{A} \to \mathcal{B}$ preserving decoherence weights and satisfying $\mathfrak{J} = \mathfrak{M} \circ \psi$ if and only if $\mathcal{B}$ is atomic, and $\mathfrak{M}$ maps the set $\mathcal{B}_0$ of atoms of $\mathcal{B}$ bijectively to $\mathfrak{J}(\mathcal{A}_0)$, and $\mathfrak{M}(0_{\mathcal{B}}) = \mathfrak{J}(0_{\mathcal{A}})$.*

Proof: " $\Longrightarrow$ ": trivial. " $\Longleftarrow$ ": $\mathfrak{M}^{-1} \circ \mathfrak{J}$ restricted to $\widetilde{\mathcal{A}} := \mathcal{A}_0 \cup \{0_{\mathcal{A}}\}$ can in an obvious way be extended to a lattice isomorphism $\psi : \mathcal{A} \to \mathcal{B}$ by requiring $\psi(\vee_{\mathcal{A}, i \in I} a_i) = \vee_{\mathcal{B}, i \in I} \psi(a_i)$ for any $\{a_i\}_{i \in I} \subset \mathcal{A}_0$. Then $\widetilde{\mathfrak{J}} := \mathfrak{M} \circ \psi$ is a positive valuation satisfying the valuation condition and extending the map $\mathfrak{N}_{|\widetilde{\mathcal{A}}}$ as required in the Definition of the allowed Boolean lattice. Since $(\mathcal{A}, \mathfrak{J})$ is an allowed Boolean lattice, $\mathfrak{J}$ is the unique positive valuation with this property and thus $\mathfrak{J} = \widetilde{\mathfrak{J}}$. That ψ preserves decoherence weights follows immediately: $d_{\varrho, \mathcal{A}}(a_1, a_2) :=$

$d_{\varrho,\mathfrak{T},k}(\mathfrak{J}(a_1),\mathfrak{J}(a_2)) = \widehat{d}_\varrho(\mathfrak{M} \circ \psi(a_1), \mathfrak{M} \circ \psi(a_2)) = \widehat{d}_\varrho(\psi(a_1), \psi(a_2))$, for all $a_1, a_2 \in \widetilde{\mathcal{A}}$, where $d_{\varrho,\mathfrak{T},k}$ denoted the decoherence functional on $\mathfrak{E}(\mathbb{H})_{2/k,\mathfrak{T}}$ (compare Remark 20 in Ref. [37]). $\quad\square$

Theorem 4 *For every allowed Boolean lattice $(\mathcal{A}, \mathfrak{J})$ in the sense of Ref. [37] there is an admissible Boolean lattice $(\mathcal{B}, \mathfrak{M})$ such that there exists an isomorphism $\psi : \mathcal{A} \to \mathcal{B}$ satisfying the conditions from Theorem 3.*

Proof: We denote by $\mathcal{A}_0$ the set of atoms of $\mathcal{A}$. We construct $\mathcal{B}$ inductively. We choose $\mathfrak{J}(\mathcal{A}_0)$ to be the set of atoms of $\mathcal{B}$ and $0_\mathcal{B} := \mathfrak{J}(0_\mathcal{A})$. We define $\mathfrak{J}(a_1) \vee_\mathcal{B} \mathfrak{J}(a_2) := \mathfrak{J}(a_1 \vee_\mathcal{A} a_2)$ for all $a_1, a_2 \in \mathcal{A}_0$. If $\mathcal{A}_0$ contains more than two elements, then $\mathfrak{J}(a_1) \vee_\mathcal{B} \mathfrak{J}(a_2) = \mathfrak{J}(a_1) \oplus \mathfrak{J}(a_2)$ for $a_1 \neq a_2$. This definition makes sense since $\mathfrak{J}(a_1) \oplus \mathfrak{J}(a_2)$ is well-defined for all $a_1, a_2 \in \mathcal{A}_0$ with $a_1 \neq a_2$ and since $\mathfrak{J}(a_1) \neq \mathfrak{J}(a_2)$ for all $a_1, a_2 \in \mathcal{A}_0$ with $a_1 \neq a_2$. If $\mathcal{A}_0$ contains exactly two elements, then $\mathfrak{J}(a_1) \vee_\mathcal{B} \mathfrak{J}(a_2) = \mathfrak{J}(a_1) \oplus \mathfrak{J}(a_2) \ominus \mathfrak{J}(0_\mathcal{A})$ for $a_1 \neq a_2$. This definition makes sense since $\mathfrak{J}(a_1) \oplus \mathfrak{J}(a_2) \ominus \mathfrak{J}(0_\mathcal{A})$ is well-defined for all $a_1, a_2 \in \mathcal{A}_0$ with $a_1 \neq a_2$ and since $\mathfrak{J}(a_1) \neq \mathfrak{J}(a_2)$ for all $a_1, a_2 \in \mathcal{A}_0$ with $a_1 \neq a_2$. $\quad\square$

The *full D-posets* also discussed in Ref. [37] are trivially contained in the class of admissible Boolean lattices defined in Definition 3.

From Theorem 4 and our Definition 5 of the implication relation between effect histories it follows immediately that if e_1 and e_2 are homogeneous effect histories such that $e_1 \Longrightarrow_\varrho e_2$ in the sense of Ref. [37], then also $e_1 \Longrightarrow_\varrho e_2$ in the sense of Definition 5.

Thus, Theorem 4 clearly shows that Rule 3 is indeed a generalization of the Rule 2 stated in Ref. [37].

IV. Summary

We now summarize our discussion by stating the general axioms for a generalized quantum theory based on our generalized history concept. This subsection parallels the discussion in Ref. [21].

1. The space $\mathfrak{U}$ of general history propositions.

   - The space $\mathfrak{U}$ carries a canonical D-poset structure denoted by $\oplus$.
     - In this work $\mathfrak{U}$ is given by $\mathfrak{E}_{fin}^\otimes(\mathbb{H})$.

2. The space $\mathcal{U}$ of history filters or homogeneous histories.

   - $\mathcal{U}$ is the space of the basic physical properties of a physical system with a direct physical interpretation. An element of $\mathcal{U}$ is a time-ordered sequence of one-time propositions about the system. There exists a map F mapping the elements of $\mathcal{U}$ to a D-poset $\mathfrak{E}$. $\mathfrak{E}$ can be interpreted as the set of (equivalence classes of) one-time propositions.

- In this work $\mathcal{U}$ equals the space of homogeneous effect histories of the first kind $\mathcal{U} = \mathbb{E}_{fin}(\mathbb{H})$, cf. Definition 1; $\mathfrak{E}$ is given by $\mathfrak{E}(\mathbb{H})$ and F is given by $F(u) = C_{t_0}(u)^\dagger C_{t_0}(u)$.

- $\mathcal{U}$ is a partially ordered set with unit history 1 and null history 0.

- There exists an order preserving map $\tau : \mathcal{U} \to \mathfrak{U}$, i.e., $\tau(\mathcal{U}) \subset \mathfrak{U}$.

  - In this work τ is given by σ_{fin}.

- $\mathcal{U}$ is a partial semigroup with composition law $\circ$, cf. Ref. [21]. $a \circ b$ is well-defined if $t_f(a) < t_i(b)$. In this case we say that a PROCEEDS b or that b FOLLOWS a. Further, $1 \circ a = a \circ 1 = a$ and $a \circ 0 = 0 \circ a = 0$. If $a \circ b$ is defined, then $a \circ b = a \wedge b$, in particular the right hand side is well-defined.

- The partial ordering on $\mathcal{U}$ induces a partial unary operation $\neg$ (complementation) and two partial binary operations $\wedge$ and $\vee$ (meet and join) on $\mathcal{U}$.

3. The space of decoherence functionals.

   - A decoherence functional is a map $d : \mathfrak{U} \times \mathfrak{U} \to \mathbb{C}$ which satisfies for all $\alpha, \alpha', \beta \in \mathfrak{U}$

     - $d(\alpha, \alpha) \in \mathbb{R}$ and $d(\alpha, \alpha) \geq 0$.
     - $d(\alpha, \beta) = d(\beta, \alpha)^*$.
     - $d(1, 1) = 1$.
     - $d(0, \alpha) = 0$, for all α.
     - $d(\alpha_1 \oplus \alpha_2, \beta) = d(\alpha_1, \beta) + d(\alpha_2, \beta)$ for all $\alpha_1, \alpha_2, \beta \in \mathfrak{U}$ for which $\alpha_1 \oplus \alpha_2$ is well-defined.

   - In Ref. [37] it was possible to explicitly construct the decoherence functional on all inhomogeneous effect histories considered. In this work we have no explicit construction of the decoherence functional on $\mathfrak{E}_{fin}^{\otimes}(\mathbb{H})$. Only its existence is known by Theorem 1.

4. The physical interpretation.

   - The physically interesting subsets of $\mathfrak{U}$ are the 'admissible' Boolean sublattices $\mathcal{B}$ of $\mathfrak{U}$ (see Definition 3) on which a positive valuation $\mathfrak{M}$ can be defined with values in $\mathfrak{U}$ such that for every $u \in \mathcal{B}$ the value $\mathfrak{M}(u)$ does not depend upon the particular 'admissible' Boolean lattice $\mathcal{B}$ chosen.

   - The map $\mathfrak{M}$ 'lifts' the lattice structure of $\mathcal{B}$ to the D-poset structure of $\mathfrak{U}$ and every decoherence functional on $\mathfrak{U}$ induces a consistency functional on $\mathcal{B}$.

   - The decoherence functional induces a probability measure on the consistent (w.r.t. the decoherence functional) 'admissible' Boolean sublattices of $\mathfrak{U}$.

   - On the 'admissible' Boolean sublattices of $\mathfrak{U}$ the decoherence functional defines a partial logical implication which allows to make logical inferences.

- The description of a physical system and reasoning in terms of elements of $\mathcal{U}$ (homogeneous effect histories of the first kind) is only indirectly possible by using the map $\tau : \mathcal{U} \rightarrow \mathfrak{U}$.

- While homogeneous effect histories have a direct physical interpretation in terms of time sequences of physical properties, inhomogeneous (effect) histories have no such direct interpretation. We tentatively suggest, however, that they may be interpreted as representatives of *unsharp quantum events*, i.e, events which cannot be associated with some fixed time, but which are smeared out in time.

Acknowledgments

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